Replica Bethe ansatz derivation of the GOE Tracy-Widom distribution in one-dimensional directed polymers with free boundary conditions

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The distribution function of the free energy fluctuations in one-dimensional directed polymers with free boundary conditions is derived by mapping the replicated problem to the N-particle quantum boson system with attractive interactions. It is shown that in the thermodynamic limit this function is described by the universal Tracy-Widom distribution of the Gaussian orthogonal ensemble.

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I. INTRODUCTION

Directed polymers in a quenched random potential have been the subject of intense investigations during the past two decades (see e.g. [1–6]). In the one-dimensional case we deal with an elastic string directed along the τ-axis within an interval $[0, t]$. Randomness enters the problem through a disorder potential $V[\phi(\tau), \tau]$, which competes against the elastic energy. The problem is defined by the Hamiltonian

$$H[\phi(\tau), V] = \int_0^t d\tau \Big\{ \frac{1}{2} \big[\partial_\tau \phi(\tau)\big]^2 + V[\phi(\tau), \tau] \Big\}; \tag{1}$$

where the disorder potential $V[\phi, \tau]$ is Gaussian distributed with a zero mean $\overline{V(\phi, \tau)} = 0$ and the δ-correlations:

$$\overline{V(\phi, \tau) V(\phi', \tau')} = u \delta(\tau - \tau') \delta(\phi - \phi') \tag{2}$$

Here the parameter u describes the strength of the disorder. Note that such system is equivalent to the problem of the Kardar-Parisi-Zang (KPZ) equation describing the growth in time of an interface in the presence of noise [7].

In what follows we consider the problem in which the polymer is fixed at the origin, $\phi(0) = 0$ and it is free at $\tau = t$. In other words, for a given realization of the random potential V the partition function of the considered system is:

$$Z = \int_{-\infty}^{+\infty} dx \, Z(x) \; = \; \exp\{-\beta F\} \tag{3}$$

where

$$Z(x) = \int_{\phi(0)=0}^{\phi(t)=x} \mathcal{D}\phi(\tau) \, \mathrm{e}^{-\beta H[\phi]} \tag{4}$$

is the partition function of the system with the fixed boundary conditions, $\phi(0) = 0$ and $\phi(t) = x$ and where F is the total free energy. Besides the usual extensive part $f_0 t$ (where f_0 is the linear free energy density), the total free energy F of such system is known to contain the disorder dependent fluctuating contribution $\tilde{F}$. In the limit of large t the typical value of the free energy fluctuations scales with t as $\tilde{F} \propto t^{1/3}$ (see e.g. [3–6]). In other words, the total free energy of the system can be represented as

$$F \; = \; f_0 t \; + \; c \, t^{1/3} \, f \tag{5}$$

where c is a non-universal parameter, which depends on the temperature and the strength of disorder, and f is the random quantity which in the thermodynamic limit $t \to \infty$ is described by a non-trivial universal distribution function $P(f)$. Note that according to eqs.(3)-(5), the trivial self-averaging contribution $f_0 t$ to the free energy can be eliminated by a simple redefinition of the partition function:

$$Z \; = \; \exp\{-\beta f_0 t\} \, \tilde{Z} \tag{6}$$

so that

$$\tilde{Z} = \exp\{-\lambda f\} \tag{7}$$

where

$$\lambda = \beta c t^{1/3} \tag{8}$$

For the similar problem with the zero boundary conditions, $\phi(0) = \phi(t) = 0$, the corresponding distribution function was proved to be described by the Gaussian Unitary Ensemble (GUE) Tracy-Widom distribution [8–11]. In the course of this proof rather efficient Bethe ansatz replica technique has been developed [10, 11]. In particular, in terms of this technique the corresponding multi-point free energy distribution functions have been derived [12]. Recently, the free energy distribution function for the directed polymer problem with the free boundary conditions, eqs.(1)-(4), has been obtained [13]. It was shown that the function $P(f)$ is the Gaussian Orthogonal Ensemble (GOE) Tracy-Widom distribution. In this paper I would like to present sufficiently simple alternative way of derivation of the same result which does not require rather complicated technique of the Fredholm Pfaffian described in [13].

Let us introduce the function

$$W(f) \equiv \int_{f}^{\infty} df' \; P(f') \tag{9}$$

which gives the probability that the random free energy is bigger that a given value f. It will be shown that in the thermodynamic limit, $t \to \infty$, this function is equal to the Fredholm determinant

$$W(f) = \det(1 - \hat{K}_{-f}) \equiv F_1(-f) \tag{10}$$

with the kernel

$$K_{-f}(\omega, \omega') = \text{Ai}(\omega + \omega' - f) \; ; \qquad (\omega, \omega' > 0) \tag{11}$$

which is the GOE Tracy-Widom distribution [14, 15]. Explicitly,

$$F_1(s) = \exp\left[-\frac{1}{2}\int_{s}^{+\infty} d\xi \; (\xi - s) \, q^2(\xi) \; - \frac{1}{2}\int_{s}^{+\infty} d\xi \; q(\xi)\right] \tag{12}$$

where $q(\xi)$ is the solution of the Panlevé II differential equation, $q''(\xi) = \xi q(\xi) + 2q^3(\xi)$, with the boundary condition $q(\xi \to +\infty) = \text{Ai}(\xi)$.

It should be noted that the present paper is rather technical. The main message of this work is not the final result itself (which is well known anyway) but the presentation of the general method and new technical tricks used in the derivation. Section II is devoted to the standard reformulation of the considered problem in terms of one-dimensional N-particle system of quantum bosons with attractive δ-interactions [6] . Here it is shown that the calculation of the the free energy probability distribution function, eq.(9), reduces to the summation over all the spectrum of eigenstates of this N-particle problem. This summation is performed in Section III, where in the thermodynamic limit, $t \to \infty$, the result, eqs.(10)-(11) is derived. The concluding remarks and as well as the key points of the calculations are listed in the final Section IV.

II. MAPPING TO QUANTUM BOSONS

In terms of the partition function $\tilde{Z}$, eq.(7), the function $W(f)$, eq.(9), can be defined as follows:

$$W(f) = \lim_{\lambda\to\infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp(\lambda N f) \; \overline{\tilde{Z}^N} \tag{13}$$

where $\overline{(...)}$ denotes the averaging over quenched disorder. Indeed, substituting here eq.(7), we have

$$\begin{aligned} W(f) &= \lim_{\lambda\to\infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \int_{-\infty}^{+\infty} df' \; P(f') \exp\{\lambda N (f - f')\} \\ &= \lim_{\lambda\to\infty} \int_{-\infty}^{+\infty} df' \; P(f') \exp\bigl[-\exp\{\lambda(f - f')\}\bigr] \\ &= \int_{-\infty}^{+\infty} df' \; P(f') \, \theta\bigl(f - f'\bigr) \end{aligned} \tag{14}$$

which coincides with the definition, eq.(9).

Later on we will see that the integration over x in the definition of the partition function, eq.(3), requires proper regularization at both limits $\pm\infty$. For that reason it is convenient to represent it in the form of two contributions:

$$Z \;=\; \int_{-\infty}^{0} dx\; Z(x) \;+\; \int_{0}^{+\infty} dx\; Z(x) \;\equiv\; Z_{(-)} \;+\; Z_{(+)} \tag{15}$$

Thus, taking into account the definition eq.(6), we get

$$\begin{aligned} W(f) \;&=\; \lim_{\lambda\to\infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp\{\lambda N f + \beta N f_0 t\} \overline{\bigl(Z_{(-)} \;+\; Z_{(+)}\bigr)^N} \\ &=\; \lim_{\lambda\to\infty} \sum_{K,L=0}^{\infty} \frac{(-1)^{K+L}}{K!\,L!} \exp\{\lambda(K+L)f + \beta(K+L)f_0 t\}\; \overline{Z_{(-)}^{K} Z_{(+)}^{L}} \\ &=\; \lim_{\lambda\to\infty} \sum_{K,L=0}^{\infty} \frac{(-1)^{K+L}}{K!\,L!} \exp\{\lambda(K+L)f + \beta(K+L)f_0 t\} \times \\ &\times\; \int_{-\infty}^{0} dx_1 ... dx_K \int_{0}^{+\infty} dy_1 ... dy_L \Psi(x_1, ..., x_K, y_L, ..., y_1; t) \end{aligned} \tag{16}$$

where

$$\Psi(x_1, ..., x_N; t) \;\equiv\; \overline{Z(x_1)\, Z(x_2) \ldots Z(x_N)} \tag{17}$$

Using the relations, eqs.(1), (2) and (4), after simple Gaussian averaging we obtain

$$\Psi(x_1, ..., x_N; t) \;=\; \prod_{a=1}^{N} \left[\int_{\phi_a(0)=0}^{\phi_a(t)=x_a} \mathcal{D}\phi_a(\tau) \right] \; \exp\bigl(-\beta H_N[\phi_1, \phi_2, ..., \phi_N]\bigr) \tag{18}$$

where

$$H_N[\phi_1, \phi_2, ..., \phi_N] \;=\; \frac{1}{2} \int_0^t d\tau \left(\sum_{a=1}^{N} \bigl[\partial_\tau \phi_a(\tau)\bigr]^2 - \beta u \sum_{a\neq b}^{N} \delta\bigl[\phi_a(\tau) - \phi_b(\tau)\bigr] \right) \tag{19}$$

The propagator $\Psi(\mathbf{x}; t)$, eq.(18), describes N trajectories $\phi_a(\tau)$ all starting at zero ($\phi_a(0) = 0$), and coming to N different points $\{x_1, ..., x_N\}$ at $\tau = t$. One can easily show that $\Psi(\mathbf{x}; t)$ can be obtained as the solution of the linear differential equation

$$\beta\, \partial_t \Psi(\mathbf{x}; t) \;=\; \frac{1}{2} \sum_{a=1}^{N} \partial_{x_a}^2 \Psi(\mathbf{x}; t) \;+\; \frac{1}{2}\, \kappa \sum_{a\neq b}^{N} \delta(x_a - x_b) \Psi(\mathbf{x}; t) \tag{20}$$

with the initial condition

$$\Psi(\mathbf{x}; 0) = \Pi_{a=1}^{N} \delta(x_a) \tag{21}$$

and the interaction parameter $\kappa = \beta^3 u$. One can easily see that Eq.(20) is the imaginary-time Schrödinger equation

$$-\, \beta\, \partial_t \Psi(\mathbf{x}; t) = \hat{H} \Psi(\mathbf{x}; t) \tag{22}$$

with the Hamiltonian

$$\hat{H} = -\frac{1}{2} \sum_{a=1}^{N} \partial_{x_a}^2 - \frac{1}{2}\, \kappa \sum_{a\neq b}^{N} \delta(x_a - x_b) \tag{23}$$

which describes N bose-particles interacting via the *attractive* two-body potential $-\kappa\delta(x)$. A generic eigenstate of such system is characterized by N momenta $\{q_a\}$ ($a = 1, ..., N$) which are splitted into M ($1 \leq M \leq N$) "clusters"

described by continuous real momenta q_α ($\alpha = 1, ..., M$) and having n_α discrete imaginary "components" (for details see [10, 16–20]):

$$q_a \;\equiv\; q_r^\alpha \;=\; q_\alpha - \frac{i\kappa}{2}(n_\alpha + 1 - 2r) \;\; ; \qquad\qquad (r = 1, ..., n_\alpha) \tag{24}$$

with the constraint

$$\sum_{\alpha=1}^{M} n_\alpha = N \tag{25}$$

A generic solution $\Psi(\mathbf{x}, t)$ of the Schrödinger equation (20) with the initial conditions, Eq.(21), can be represented in the form of the linear combination of the eigenfunctions $\Psi_{\mathbf{q}}^{(M)}(\mathbf{x})$:

$$\Psi(x_1, ..., x_N; t) = \sum_{M=1}^{N} \frac{1}{M!} \Bigg[\int \mathcal{D}^{(M)}(\mathbf{q}, \mathbf{n}) \Bigg] \, |C_M(\mathbf{q}, \mathbf{n})|^2 \; \Psi_{\mathbf{q}}^{(M)}(\mathbf{x}) \Psi_{\mathbf{q}}^{(M)*}(\mathbf{0}) \; \exp\{-E_M(\mathbf{q})t\} \tag{26}$$

where we have introduced the notation

$$\int \mathcal{D}^{(M)}(\mathbf{q}, \mathbf{n}) \equiv \prod_{\alpha=1}^{M} \Bigg[\int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \Bigg] \boldsymbol{\delta}\Big(\sum_{\alpha=1}^{M} n_\alpha \; , \; N \Big) \tag{27}$$

and $\boldsymbol{\delta}(k, m)$ is the Kronecker symbol; note that the presence of this Kronecker symbol in the above equation allows to extend the summations over n_α's to infinity. Here (non-normalized) eigenfunctions are [10, 20]

$$\Psi_{\mathbf{q}}^{(M)}(\mathbf{x}) = \sum_{\mathcal{P}} \prod_{a<b}^{N} \Bigg[1 + i\kappa \frac{\mathrm{sgn}(x_a - x_b)}{q_{\mathcal{P}_a} - q_{\mathcal{P}_b}} \Bigg] \; \exp\Big[i \sum_{a=1}^{N} q_{\mathcal{P}_a} x_a \Big] \tag{28}$$

where the summation goes over $N!$ permutations $\mathcal{P}$ of N momenta q_a, eq.(24), over N particles x_a; the normalization factor

$$|C_M(\mathbf{q}, \mathbf{n})|^2 = \frac{\kappa^N}{N! \prod_{\alpha=1}^{M} (\kappa n_\alpha)} \prod_{\alpha<\beta}^{M} \frac{\big| q_\alpha - q_\beta - \frac{i\kappa}{2}(n_\alpha - n_\beta) \big|^2}{\big| q_\alpha - q_\beta - \frac{i\kappa}{2}(n_\alpha + n_\beta) \big|^2} \tag{29}$$

and the eigenvalues:

$$E_M(\mathbf{q}) \;=\; \frac{1}{2\beta} \sum_{\alpha=1}^{N} q_a^2 \;=\; \frac{1}{2\beta} \sum_{\alpha=1}^{M} \; n_\alpha q_\alpha^2 - \frac{\kappa^2}{24\beta} \sum_{\alpha=1}^{M} (n_\alpha^3 - n_\alpha) \tag{30}$$

Note that the eigenfunctions, eq.(28), are symmetric with respect to permutations of all its arguments $x_1, ..., x_N$, and

$$\Psi_{\mathbf{q}}^{(M)}(\mathbf{0}) = N! \tag{31}$$

In this way the problem of the calculation of the free energy probability distribution function, eq.(16), reduces to the summation over all the spectrum of the eigenstates of the N-particle bosonic problem, which is parametrized by the set of both the continuous, $\{q_1, ..., q_M\}$, and the discrete $\{n_1, ..., n_M\}$; $(M = 1, ..., N)$; $(N = 1, ..., \infty)$ degrees of freedom.

III. FREE ENERGY PROBABILITY DISTRIBUTION FUNCTION

Substituting eqs.(26)-(31) into eq.(16), (defining $f_0 = \frac{1}{24}\beta^4 u^2$, the factor f_0 drops out of the further calculations) we get:

$$\begin{aligned} W(f) \;=\; & 1 + \lim_{\lambda\to\infty} \sum_{K+L\geq 1}^{\infty} \; (-1)^{K+L} \; \mathrm{e}^{\lambda(K+L)f} \; \times \\ & \times \; \sum_{M=1}^{K+L} \frac{1}{M!} \prod_{\alpha=1}^{M} \Bigg[\sum_{n_\alpha=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi\kappa n_\alpha} \kappa^{n_\alpha} \mathrm{e}^{-\frac{t}{2\beta} n_\alpha q_\alpha^2 + \frac{\kappa^2}{24\beta} n_\alpha^3} \Bigg] \; \boldsymbol{\delta}\Big(\sum_{\alpha=1}^{M} n_\alpha \; , \; N \Big) \; |\tilde{C}_M(\mathbf{q}, \mathbf{n})|^2 \; I_{K,L}(\mathbf{q}, \mathbf{n}) \end{aligned} \tag{32}$$

where

$$|\tilde{C}_M(\mathbf{q},\mathbf{n})|^2 \;=\; \prod_{\alpha<\beta}^{M} \frac{\left|q_\alpha - q_\beta - \frac{i\kappa}{2}(n_\alpha - n_\beta)\right|^2}{\left|q_\alpha - q_\beta - \frac{i\kappa}{2}(n_\alpha + n_\beta)\right|^2} \tag{33}$$

and

$$\begin{aligned} I_{K,L}(\mathbf{q},\mathbf{n}) \;=\; & \sum_{\mathcal{P}^{(K,L)}} \sum_{\mathcal{P}^{(K)}} \sum_{\mathcal{P}^{(L)}} \prod_{a=1}^{K} \prod_{c=1}^{L} \left[\frac{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}} - i\kappa}{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}}}\right] \times \prod_{a<b}^{K} \left[\frac{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_b^{(K)}} - i\kappa}{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_b^{(K)}}}\right] \times \prod_{c<d}^{L} \left[\frac{q_{\mathcal{P}_c^{(L)}} - q_{\mathcal{P}_d^{(L)}} + i\kappa}{q_{\mathcal{P}_c^{(L)}} - q_{\mathcal{P}_d^{(L)}}}\right] \times \\ & \times \int_{-\infty<x_1\leq\ldots\leq x_K\leq 0} dx_1\ldots dx_K \;\exp\Big[i\sum_{a=1}^{K}(q_{\mathcal{P}_a^{(K)}} - i\epsilon)x_a\Big] \\ & \times \int_{0\leq y_L\leq\ldots\leq y_1<+\infty} dy_L\ldots dy_1 \;\exp\Big[i\sum_{c=1}^{L}(q_{\mathcal{P}_c^{(L)}} + i\epsilon)y_c\Big] \end{aligned} \tag{34}$$

Here the summation over all permutations $\mathcal{P}$ of $(K+L)$ momenta $\{q_1,\ldots,q_{K+L}\}$ over K "negative" particles $\{x_1,\ldots,x_K\}$ and L "positive" particles $\{y_L,\ldots,y_1\}$ are divided into three parts: the permutations $\mathcal{P}^{(K)}$ of K momenta (taken at random out of the total list $\{q_1,\ldots,q_{K+L}\}$) over K "negative" particles, the permutations $\mathcal{P}^{(L)}$ of the remaining L momenta over L "positive" particles, and finally the permutations $\mathcal{P}^{(K,L)}$ (or the exchange) of the momenta between the group "K" and the group "L". Note also that the integrations both over x_a's and over y_c's in eq.(34) require proper regularization at $-\infty$ and $+\infty$ correspondingly. This is done in the standard way by introducing a supplementary parameter ϵ which will be set to zero in final results. The result of the integrations can be represented as follows:

$$\begin{aligned} I_{K,L}(\mathbf{q},\mathbf{n}) \;=\; & i^{-(K+L)} \sum_{\mathcal{P}^{(K,L)}} \prod_{a=1}^{K} \prod_{c=1}^{L} \left[\frac{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}} - i\kappa}{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}}}\right] \times \\ & \times \sum_{\mathcal{P}^{(K)}} \frac{1}{q^{(-)}_{\mathcal{P}_1^{(K)}}\big(q^{(-)}_{\mathcal{P}_1^{(K)}} + q^{(-)}_{\mathcal{P}_2^{(K)}}\big)\ldots\big(q^{(-)}_{\mathcal{P}_1^{(K)}} + \ldots + q^{(-)}_{\mathcal{P}_K^{(K)}}\big)} \prod_{a<b}^{K} \left[\frac{q^{(-)}_{\mathcal{P}_a^{(K)}} - q^{(-)}_{\mathcal{P}_b^{(K)}} - i\kappa}{q^{(-)}_{\mathcal{P}_a^{(K)}} - q^{(-)}_{\mathcal{P}_b^{(K)}}}\right] \times \\ & \times \sum_{\mathcal{P}^{(L)}} \frac{(-1)^L}{q^{(+)}_{\mathcal{P}_1^{(L)}}\big(q^{(+)}_{\mathcal{P}_1^{(L)}} + q^{(+)}_{\mathcal{P}_2^{(L)}}\big)\ldots\big(q^{(+)}_{\mathcal{P}_1^{(L)}} + \ldots + q^{(+)}_{\mathcal{P}_L^{(L)}}\big)} \prod_{c<d}^{L} \left[\frac{q^{(+)}_{\mathcal{P}_c^{(L)}} - q^{(+)}_{\mathcal{P}_d^{(L)}} + i\kappa}{q^{(+)}_{\mathcal{P}_c^{(L)}} - q^{(+)}_{\mathcal{P}_d^{(L)}}}\right] \end{aligned} \tag{35}$$

where

$$q_a^{(\pm)} \;\equiv\; q_a \pm i\epsilon \tag{36}$$

Using the "magic" Bethe ansatz combinatorial identity [13],

$$\sum_{P} \frac{1}{q_{p_1}(q_{p_1}+q_{p_2})\ldots(q_{p_1}+\ldots+q_{p_N})} \prod_{a<b}^{N} \left[\frac{q_{p_a} - q_{p_b} - i\kappa}{q_{p_a} - q_{p_b}}\right] \;=\; \frac{1}{\prod_{a=1}^{N} q_a} \prod_{a<b}^{N} \left[\frac{q_a + q_b + i\kappa}{q_a + q_b}\right] \tag{37}$$

(where the summation goes over all permutations P of N momenta $\{q_1,\ldots,q_N\}$) we get:

$$\begin{aligned} I_{K,L}(\mathbf{q},\mathbf{n}) \;=\; & i^{-(K+L)} \sum_{\mathcal{P}^{(K,L)}} \prod_{a=1}^{K} \prod_{c=1}^{L} \left[\frac{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}} - i\kappa}{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}}}\right] \times \\ & \times \frac{1}{\prod_{a=1}^{K} q^{(-)}_{\mathcal{P}_a^{(K)}}} \prod_{a<b}^{K} \left[\frac{q^{(-)}_{\mathcal{P}_a^{(K)}} + q^{(-)}_{\mathcal{P}_b^{(K)}} + i\kappa}{q^{(-)}_{\mathcal{P}_a^{(K)}} + q^{(-)}_{\mathcal{P}_b^{(K)}}}\right] \times \frac{(-1)^L}{\prod_{c=1}^{L} q^{(+)}_{\mathcal{P}_c^{(L)}}} \prod_{c<d}^{L} \left[\frac{q^{(+)}_{\mathcal{P}_c^{(L)}} + q^{(+)}_{\mathcal{P}_d^{(L)}} - i\kappa}{q^{(+)}_{\mathcal{P}_c^{(L)}} + q^{(+)}_{\mathcal{P}_d^{(L)}}}\right] \end{aligned} \tag{38}$$

Further simplification comes from one important property of the Bethe ansatz wave function, eq.(28). It has such structure that for ordered particles positions (e.g. $x_1 < x_2 < \ldots < x_N$) in the summation over permutations

the momenta q_a belonging to the same cluster also remain ordered. In other words, if we consider the momenta, eq.(24), of a cluster α, $\{q_1^\alpha, q_2^\alpha, ..., q_{n_\alpha}^\alpha\}$, belonging correspondingly to the particles $\{x_{i_1} < x_{i_2} < ... < x_{i_{n_\alpha}}\}$, the permutation of any two momenta q_r^α and $q_{r'}^\alpha$ of this *ordered* set gives zero contribution. Thus, in order to perform the summation over the permutations $\mathcal{P}^{(K,L)}$ in eq.(38) it is sufficient to split the momenta of each cluster into two parts: $\{q_1^\alpha, ..., q_{m_\alpha}^\alpha || q_{m_\alpha+1}^\alpha ..., q_{n_\alpha}^\alpha\}$, where $m_\alpha = 0, 1, ..., n_\alpha$ and where the momenta $q_1^\alpha, ..., q_{m_\alpha}^\alpha$ belong to the particles of the sector "K", while the momenta $q_{m_\alpha+1}^\alpha ..., q_{n_\alpha}^\alpha$ belong to the particles of the sector "L".

Let us introduce the numbering of the momenta of the sector "L" in the reversed order:

$$
\begin{array}{rcl}
q_{n_\alpha}^\alpha & \to & {q^*}_1^\alpha \\
q_{n_\alpha-1}^\alpha & \to & {q^*}_2^\alpha \\
 & .... & \\
q_{m_\alpha+1}^\alpha & \to & {q^*}_{s_\alpha}^\alpha
\end{array}
\tag{39}
$$

where $m_\alpha + s_\alpha = n_\alpha$ and (s.f. eq.(24))

$$
{q^*}_r^\alpha \; = \; q_\alpha + \frac{i\kappa}{2}(n_\alpha + 1 - 2r) \; = \; q_\alpha + \frac{i\kappa}{2}(m_\alpha + s_\alpha + 1 - 2r)
\tag{40}
$$

By definition, the integer parameters $\{m_\alpha\}$ and $\{s_\alpha\}$ fulfill the global constrains

$$
\sum_{\alpha=1}^{M} m_\alpha \; = \; K
\tag{41}
$$

$$
\sum_{\alpha=1}^{M} s_\alpha \; = \; L
\tag{42}
$$

In this way the summation over permutations $\mathcal{P}^{(K,L)}$ in eq.(38) is changed by the summations over the integer parameters $\{m_\alpha\}$ and $\{s_\alpha\}$:

$$
\sum_{\mathcal{P}^{(K,L)}} (...) \; \to \; \prod_{\alpha=1}^{M} \Biggl[\sum_{m_\alpha+s_\alpha\geq 1}^{\infty} \boldsymbol{\delta}\Bigl(m_\alpha + s_\alpha \; , \; n_\alpha\Bigr)\Biggr] \; \boldsymbol{\delta}\Bigl(\sum_{\alpha=1}^{M} m_\alpha \; , \; K\Bigr) \; \boldsymbol{\delta}\Bigl(\sum_{\alpha=1}^{M} s_\alpha \; , \; L\Bigr) \; (...)
\tag{43}
$$

which allows to lift the summations over K, L, and $\{n_\alpha\}$ in eq.(32). In terms of the parameters $\{m_\alpha\}$ and $\{s_\alpha\}$ the product factors in eq.(38) are expressed as follows:

$$
\prod_{a=1}^{K} q^{(-)}_{\mathcal{P}_a^{(K)}} \; = \; \prod_{\alpha=1}^{M} \prod_{r=1}^{m_\alpha} q_r^{\alpha\,(-)}
\tag{44}
$$

$$
\prod_{a=1}^{L} q^{(+)}_{\mathcal{P}_a^{(L)}} \; = \; \prod_{\alpha=1}^{M} \prod_{r=1}^{s_\alpha} {q^*}_r^{\alpha\,(+)}
\tag{45}
$$

$$
\prod_{a<b}^{K} \left[\frac{q^{(-)}_{\mathcal{P}_a^{(K)}} + q^{(-)}_{\mathcal{P}_b^{(K)}} + i\kappa}{q^{(-)}_{\mathcal{P}_a^{(K)}} + q^{(-)}_{\mathcal{P}_b^{(K)}}} \right] \; = \; \prod_{\alpha=1}^{M} \prod_{1\leq r<r'}^{m_\alpha} \left[\frac{q_r^{\alpha\,(-)} + q_{r'}^{\alpha\,(-)} + i\kappa}{q_r^{\alpha\,(-)} + q_{r'}^{\alpha\,(-)}} \right] \times \prod_{1\leq\alpha<\beta}^{M} \prod_{r=1}^{m_\alpha} \prod_{r'=1}^{m_\beta} \left[\frac{q_r^{\alpha\,(-)} + q_{r'}^{\beta\,(-)} + i\kappa}{q_r^{\alpha\,(-)} + q_{r'}^{\beta\,(-)}} \right]
\tag{46}
$$

$$
\prod_{c<d}^{L} \left[\frac{q^{(+)}_{\mathcal{P}_c^{(L)}} + q^{(+)}_{\mathcal{P}_d^{(L)}} - i\kappa}{q^{(+)}_{\mathcal{P}_c^{(L)}} + q^{(+)}_{\mathcal{P}_d^{(L)}}} \right] \; = \; \prod_{\alpha=1}^{M} \prod_{1\leq r<r'}^{s_\alpha} \left[\frac{{q^*}_r^{\alpha\,(+)} + {q^*}_{r'}^{\alpha\,(+)} - i\kappa}{{q^*}_r^{\alpha\,(+)} + {q^*}_{r'}^{\alpha\,(+)}} \right] \times \prod_{1\leq\alpha<\beta}^{M} \prod_{r=1}^{s_\alpha} \prod_{r'=1}^{s_\beta} \left[\frac{{q^*}_r^{\alpha\,(+)} + {q^*}_{r'}^{\beta\,(+)} - i\kappa}{{q^*}_r^{\alpha\,(+)} + {q^*}_{r'}^{\beta\,(+)}} \right]
\tag{47}
$$

$$
\prod_{a=1}^{K} \prod_{c=1}^{L} \left[\frac{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}} - i\kappa}{q_{\mathcal{P}_a^{(K)}} - q_{\mathcal{P}_c^{(L)}}} \right] \; = \; \prod_{1\leq\alpha<\beta}^{M} \left\{ \prod_{r=1}^{m_\alpha} \prod_{r'=1}^{s_\beta} \left[\frac{q_r^\alpha - {q^*}_{r'}^\beta - i\kappa}{q_r^\alpha + {q^*}_{r'}^\beta} \right] \times \prod_{r=1}^{s_\alpha} \prod_{r'=1}^{m_\beta} \left[\frac{{q^*}_r^\alpha - q_{r'}^\beta - i\kappa}{{q^*}_r^\alpha - q_{r'}^\beta} \right] \right\} \times
$$

$$
\times \prod_{\alpha=1}^{M} \prod_{r=1}^{m_\alpha} \prod_{r'=1}^{s_\alpha} \left[\frac{q_r^\alpha - {q^*}_{r'}^\alpha - i\kappa}{q_r^\alpha - {q^*}_{r'}^\alpha} \right]
\tag{48}
$$

Substituting eqs.(43)-(48) into eq.(38), and then substituting the resulting expression into eq.(32) we obtain

$$W(f) = \lim_{\lambda\to\infty}\Bigg\{1+\sum_{M=1}^{\infty}\frac{(-1)^M}{M!}\prod_{\alpha=1}^{M}\Bigg[\sum_{m_\alpha+s_\alpha\geq 1}^{\infty}(-1)^{m_\alpha+s_\alpha-1}\int_{-\infty}^{+\infty}dq_\alpha\,\frac{\mathcal{G}\big(q_\alpha,m_\alpha,s_\alpha\big)}{2\pi\kappa(m_\alpha+s_\alpha)}\times$$
$$\times\;\mathrm{e}^{-\frac{t}{2\beta}(m_\alpha+s_\alpha)q_\alpha^2+\frac{\kappa^2}{24\beta}(m_\alpha+s_\alpha)^3+\lambda(m_\alpha+s_\alpha)f}\Bigg]\,|\tilde{C}_M(\mathbf{q},\mathbf{m}+\mathbf{s})|^2\prod_{1\leq\alpha<\beta}^{M}\mathcal{G}_{\alpha\beta}\big(\mathbf{q},\mathbf{m},\mathbf{s}\big)\Bigg\} \tag{49}$$

where

$$|\tilde{C}_M(\mathbf{q},\mathbf{m}+\mathbf{s})|^2 = \prod_{\alpha<\beta}^{M}\frac{\big|q_\alpha-q_\beta-\frac{i\kappa}{2}(m_\alpha+s_\alpha-m_\beta-s_\beta)\big|^2}{\big|q_\alpha-q_\beta-\frac{i\kappa}{2}(m_\alpha+s_\alpha+m_\beta+s_\beta)\big|^2} \tag{50}$$

$$\mathcal{G} = \frac{(-1)^{s_\alpha}(-i\kappa)^{(m_\alpha+s_\alpha)}}{\prod_{r=1}^{m_\alpha}q_r^{\alpha\,(-)}\prod_{r=1}^{s_\alpha}q^{*\alpha\,(+)}_{\;r}}\prod_{r<r'}^{m_\alpha}\left[\frac{q_r^{\alpha\,(-)}+q_{r'}^{\alpha\,(-)}+i\kappa}{q_r^{\alpha\,(-)}+q_{r'}^{\alpha\,(-)}}\right]\prod_{r<r'}^{s_\alpha}\left[\frac{q^{*\alpha\,(+)}_{\;r}+q^{*\alpha\,(+)}_{\;r'}-i\kappa}{q^{*\alpha\,(+)}_{\;r}+q^{*\alpha\,(+)}_{\;r'}}\right]\prod_{r=1}^{m_\alpha}\prod_{r'=1}^{s_\alpha}\left[\frac{q_r^\alpha-q^{*\alpha}_{\;r'}-i\kappa}{q_r^\alpha-q^{*\alpha}_{\;r'}}\right] \tag{51}$$

and

$$\mathcal{G}_{\alpha\beta} = \prod_{r=1}^{m_\alpha}\prod_{r'=1}^{m_\beta}\left[\frac{q_r^{\alpha\,(-)}+q_{r'}^{\beta\,(-)}+i\kappa}{q_r^{\alpha\,(-)}+q_{r'}^{\beta\,(-)}}\right]\prod_{r=1}^{s_\alpha}\prod_{r'=1}^{s_\beta}\left[\frac{q^{*\alpha\,(+)}_{\;r}+q^{*\beta\,(+)}_{\;r'}-i\kappa}{q^{*\alpha\,(+)}_{\;r}+q^{*\beta\,(+)}_{\;r'}}\right]\prod_{r=1}^{m_\alpha}\prod_{r'=1}^{s_\beta}\left[\frac{q_r^\alpha-q^{*\beta}_{\;r'}-i\kappa}{q_r^\alpha+q^{*\beta}_{\;r'}}\right]\times\prod_{r=1}^{s_\alpha}\prod_{r'=1}^{m_\beta}\left[\frac{q^{*\alpha}_{\;r}-q_{r'}^\beta-i\kappa}{q^{*\alpha}_{\;r}-q_{r'}^\beta}\right] \tag{52}$$

The product factors in eq.(51) can be easily expressed it terms of the Gamma functions:

$$\prod_{r=1}^{m_\alpha}q_r^{\alpha\,(-)} = \prod_{r=1}^{m_\alpha}\Big[q_\alpha^{\,(-)}-\frac{i\kappa}{2}(m_\alpha+s_\alpha+1)+i\kappa r\Big] = (i\kappa)^{m_\alpha}\frac{\Gamma\Big(\frac{1}{2}-\frac{s_\alpha-m_\alpha}{2}-\frac{iq_\alpha^{\,(-)}}{\kappa}\Big)}{\Gamma\Big(\frac{1}{2}-\frac{s_\alpha+m_\alpha}{2}-\frac{iq_\alpha^{\,(-)}}{\kappa}\Big)} \tag{53}$$

$$\prod_{r=1}^{s_\alpha}q^{*\alpha\,(+)}_{\;r} = \prod_{r=1}^{s_\alpha}\Big[q_\alpha^{\,(+)}+\frac{i\kappa}{2}(m_\alpha+s_\alpha+1)-i\kappa r\Big] = (-i\kappa)^{s_\alpha}\frac{\Gamma\Big(\frac{1}{2}-\frac{m_\alpha-s_\alpha}{2}+\frac{iq_\alpha^{\,(+)}}{\kappa}\Big)}{\Gamma\Big(\frac{1}{2}-\frac{m_\alpha+s_\alpha}{2}+\frac{iq_\alpha^{\,(+)}}{\kappa}\Big)} \tag{54}$$

$$\prod_{r<r'}^{m_\alpha}\left[\frac{q_r^{\alpha\,(-)}+q_{r'}^{\alpha\,(-)}+i\kappa}{q_r^{\alpha\,(-)}+q_{r'}^{\alpha\,(-)}}\right] = 2^{-(m_\alpha-1)}\frac{\Gamma\Big(m_\alpha-s_\alpha-\frac{2iq_\alpha^{\,(-)}}{\kappa}\Big)\Gamma\Big(1-\frac{m_\alpha+s_\alpha}{2}-\frac{iq_\alpha^{\,(-)}}{\kappa}\Big)}{\Gamma\Big(\frac{m_\alpha-s_\alpha}{2}-\frac{iq_\alpha^{\,(-)}}{\kappa}\Big)\Gamma\Big(1-s_\alpha-\frac{2iq_\alpha^{\,(-)}}{\kappa}\Big)} \tag{55}$$

$$\prod_{r<r'}^{s_\alpha}\left[\frac{q^{*\alpha\,(+)}_{\;r}+q^{*\alpha\,(+)}_{\;r'}-i\kappa}{q^{*\alpha\,(+)}_{\;r}+q^{*\alpha\,(+)}_{\;r'}}\right] = 2^{-(s_\alpha-1)}\frac{\Gamma\Big(s_\alpha-m_\alpha+\frac{2iq_\alpha^{\,(+)}}{\kappa}\Big)\Gamma\Big(1-\frac{m_\alpha+s_\alpha}{2}+\frac{iq_\alpha^{\,(+)}}{\kappa}\Big)}{\Gamma\Big(\frac{s_\alpha-m_\alpha}{2}+\frac{iq_\alpha^{\,(+)}}{\kappa}\Big)\Gamma\Big(1-m_\alpha+\frac{2iq_\alpha^{\,(+)}}{\kappa}\Big)} \tag{56}$$

$$\prod_{r=1}^{m_\alpha}\prod_{r'=1}^{s_\alpha}\left[\frac{q_r^\alpha-q^{*\alpha}_{\;r'}-i\kappa}{q_r^\alpha-q^{*\alpha}_{\;r'}}\right] = \frac{\Gamma\big(1+m_\alpha+s_\alpha\big)}{\Gamma\big(1+m_\alpha\big)\Gamma\big(1+s_\alpha\big)} \tag{57}$$

Substituting the above expressions into eq.(51) and using the standard relations for the Gamma functions,

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{58}$$

$$\Gamma(1+z) = z\,\Gamma(z) \tag{59}$$

$$\Gamma\Big(\frac{1}{2}+z\Big) = \frac{\sqrt{\pi}\,\Gamma\big(1+2z\big)}{2^{2z}\,\Gamma\big(1+z\big)} \tag{60}$$

for the factor $\mathcal{G}$, eq.(51), we get

$$\mathcal{G}\big(q_\alpha, m_\alpha, s_\alpha\big) \;=\; \frac{\Gamma\Big(s_\alpha + \frac{2i}{\kappa}q_\alpha{}^{(-)}\Big)\,\Gamma\Big(m_\alpha - \frac{2i}{\kappa}q_\alpha{}^{(+)}\Big)\,\Gamma\big(1+m_\alpha+s_\alpha\big)}{2^{(m_\alpha+s_\alpha)}\Gamma\Big(m_\alpha+s_\alpha+\frac{2i}{\kappa}q_\alpha{}^{(-)}\Big)\,\Gamma\Big(m_\alpha+s_\alpha-\frac{2i}{\kappa}q_\alpha{}^{(+)}\Big)\,\Gamma\big(1+m_\alpha\big)\Gamma\big(1+s_\alpha\big)} \tag{61}$$

Similar calculations for the factor $\mathcal{G}_{\alpha\beta}$ yield the following expression

$$\mathcal{G}_{\alpha\beta}\big(\mathbf{q},\mathbf{m},\mathbf{s}\big) \;=\; \frac{\Gamma\Big[1+\frac{m_\alpha+m_\beta-s_\alpha-s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha{}^{(-)}+q_\beta{}^{(-)}\big)\Big]\,\Gamma\Big[1-\frac{m_\alpha+m_\beta+s_\alpha+s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha{}^{(-)}+q_\beta{}^{(-)}\big)\Big]}{\Gamma\Big[1-\frac{m_\alpha-m_\beta+s_\alpha+s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha{}^{(-)}+q_\beta{}^{(-)}\big)\Big]\,\Gamma\Big[1+\frac{m_\alpha-m_\beta-s_\alpha-s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha{}^{(-)}+q_\beta{}^{(-)}\big)\Big]}\times$$

$$\times\frac{\Gamma\Big[1-\frac{m_\alpha+m_\beta-s_\alpha-s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha{}^{(+)}+q_\beta{}^{(+)}\big)\Big]\,\Gamma\Big[1-\frac{m_\alpha+m_\beta+s_\alpha+s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha{}^{(+)}+q_\beta{}^{(+)}\big)\Big]}{\Gamma\Big[1-\frac{m_\alpha+m_\beta+s_\alpha-s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha{}^{(+)}+q_\beta{}^{(+)}\big)\Big]\,\Gamma\Big[1-\frac{m_\alpha+m_\beta-s_\alpha+s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha{}^{(+)}+q_\beta{}^{(+)}\big)\Big]}\times$$

$$\times\frac{\Gamma\Big[1+\frac{m_\alpha+m_\beta+s_\alpha+s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]\,\Gamma\Big[1+\frac{-m_\alpha+m_\beta+s_\alpha-s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]}{\Gamma\Big[1+\frac{-m_\alpha+m_\beta+s_\alpha+s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]\,\Gamma\Big[1+\frac{m_\alpha+m_\beta+s_\alpha-s_\beta}{2} + \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]}\times$$

$$\times\frac{\Gamma\Big[1+\frac{m_\alpha+m_\beta+s_\alpha+s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]\,\Gamma\Big[1+\frac{m_\alpha-m_\beta-s_\alpha+s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]}{\Gamma\Big[1+\frac{m_\alpha+m_\beta-s_\alpha+s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]\,\Gamma\Big[1+\frac{m_\alpha-m_\beta+s_\alpha+s_\beta}{2} - \frac{i}{\kappa}\big(q_\alpha-q_\beta\big)\Big]} \tag{62}$$

Redefining

$$q_\alpha \;=\; \frac{\kappa}{2\lambda}\,p_\alpha \tag{63}$$

with

$$\lambda \;=\; \frac{1}{2}\Big(\frac{\kappa^2 t}{\beta}\Big)^{1/3} \;=\; \frac{1}{2}\,\big(\beta^5 u^2 t\big)^{1/3} \tag{64}$$

the normalization factor $|\tilde{C}_M(\mathbf{q},\mathbf{m}+\mathbf{s})|^2$, eq.(50), can be represented as follows:

$$|\tilde{C}_M(\mathbf{q},\mathbf{m}+\mathbf{s})|^2 \;=\; \prod_{\alpha<\beta}^{M}\frac{\big|\lambda\big(m_\alpha+s_\alpha\big)-\lambda\big(m_\beta+s_\beta\big)-ip_\alpha+ip_\beta\big|^2}{\big|\lambda\big(m_\alpha+s_\alpha\big)+\lambda\big(m_\beta+s_\beta\big)-ip_\alpha+ip_\beta\big|^2} \;=$$

$$=\;\prod_{\alpha=1}^{M}\big[2\lambda\big(m_\alpha+s_\alpha\big)\big]\times\det\Bigg[\frac{1}{\lambda\big(m_\alpha+s_\alpha\big)-ip_\alpha+\lambda\big(m_\beta+s_\beta\big)+ip_\beta}\Bigg]_{\alpha,\beta=1,\ldots,M} \tag{65}$$

where we have used the Cauchy double alternant identity

$$\frac{\prod_{\alpha<\beta}^{M}(a_\alpha-a_\beta)(b_\alpha-b_\beta)}{\prod_{\alpha,\beta=1}^{M}(a_\alpha-b_\beta)} \;=\; (-1)^{M(M-1)/2}\det\Big[\frac{1}{a_\alpha-b_\beta}\Big]_{\alpha,\beta=1,\ldots M} \tag{66}$$

with $a_\alpha = p_\alpha - i\lambda\big(m_\alpha+s_\alpha\big)$ and $b_\alpha = p_\alpha + i\lambda\big(m_\beta+s_\beta\big)$.

After rescaling, eq.(63), for the exponential factor in eq.(49) we find

$$-\frac{t}{2\beta}(m_\alpha+s_\alpha)q_\alpha^2+\frac{\kappa^2}{24\beta}(m_\alpha+s_\alpha)^3+\lambda(m_\alpha+s_\alpha)f \;=\; -\lambda(m_\alpha+s_\alpha)p_\alpha^2+\frac{1}{3}\lambda^3(m_\alpha+s_\alpha)^3+\lambda(m_\alpha+s_\alpha)f \tag{67}$$

The cubic exponential term can be linearized using the Airy function relation

$$\exp\Big[\frac{1}{3}\lambda^3 (m_\alpha + s_\alpha)^3\Big] \; = \; \int_{-\infty}^{+\infty} dy_\alpha \; \mathrm{Ai}(y_\alpha) \; \exp\Big[\lambda (m_\alpha + s_\alpha)\, y_\alpha\Big] \tag{68}$$

Substituting eqs.(68),(67) and (65) into eq.(49), and redefining $y_\alpha \; \to \; y_\alpha + p_\alpha^2 - f$, we get

$$\begin{aligned} W(f) \; = \; \lim_{\lambda\to\infty} \Bigg\{ \; 1 \; + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{\alpha=1}^{M} \Bigg[\int\int_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{2\pi} \, \mathrm{Ai}\big(y_\alpha + p_\alpha^2 - f\big) \times \\ \times \sum_{m_\alpha + s_\alpha \geq 1}^{\infty} (-1)^{m_\alpha + s_\alpha - 1} \exp\{\lambda (m_\alpha + s_\alpha) y_\alpha\} \, \mathcal{G}\Big(\frac{p_\alpha}{\lambda}, \; m_\alpha, \; s_\alpha\Big) \; 2^{m_\alpha + s_\alpha} \Bigg] \times \\ \times \; \det \hat{K}\big[(\lambda m_\alpha, \; \lambda s_\alpha, \; p_\alpha); (\lambda m_\beta, \; \lambda s_\beta, \; p_\beta)\big]_{\alpha,\beta=1,\ldots,M} \times \prod_{1\leq\alpha<\beta}^{M} \mathcal{G}_{\alpha\beta}\Big(\frac{\mathbf{p}}{\lambda}, \; \mathbf{m}, \; \mathbf{s}\Big) \Bigg\} \end{aligned} \tag{69}$$

where

$$\hat{K}\big[(\lambda m, \; \lambda s, \; p); (\lambda m', \; \lambda s', \; p')\big] \; = \; \frac{1}{\lambda m + \lambda s - ip + \lambda m' + \lambda s' + ip'} \tag{70}$$

The crucial point of the further calculations is the procedure of taking the thermodynamic limit $\lambda \to \infty$. First of all one can easily note that according to eqs.(63) and (69), it is the parameters $p_\alpha \sim \lambda q_\alpha$ which remain finite in the limit $\lambda \to \infty$. In other words, all the parameters q_α which are not multiplied by λ (e.g. in the expressions for $\mathcal{G}_\alpha$ and $\mathcal{G}_{\alpha\beta}$, eqs(61) and (62)) have to be taken to zero in this limit. Simultaneously, the summations over $\{m_\alpha\}$ and $\{s_\alpha\}$ have to be performed. The general algorithm of such summation is in the following. Let us consider the example of the sum of a general type:

$$R(\mathbf{y}, \mathbf{p}) \; = \; \lim_{\lambda\to\infty} \prod_{\alpha=1}^{M} \Bigg[\sum_{n_\alpha=1}^{\infty} (-1)^{n_\alpha - 1} \exp\{\lambda n_\alpha y_\alpha\} \Bigg] \; \Phi\Big[\mathbf{p}, \; \frac{\mathbf{p}}{\lambda}, \; \lambda\mathbf{n}; \; \mathbf{n}\Big] \tag{71}$$

where Φ is a function which depend both of λn_α's and n_α's (which are not multiplied by λ). The summations in the above example can be represented in terms of the integrals in the complex plane:

$$R(\mathbf{y}, \mathbf{p}) \; = \; \lim_{\lambda\to\infty} \prod_{\alpha=1}^{M} \Bigg[\frac{1}{2i} \int_{\mathcal{C}} \frac{dz_\alpha}{\sin(\pi z_\alpha)} \exp\{\lambda z_\alpha y_\alpha\} \Bigg] \; \Phi\Big[\mathbf{p}, \; \frac{\mathbf{p}}{\lambda}, \; \lambda\mathbf{z}; \; \mathbf{z}\Big] \tag{72}$$

where the integration goes over the contour $\mathcal{C}$ shown in Fig.1(a). Shifting the contour to the position $\mathcal{C}'$ shown in Fig.1(b) (assuming that there is no contribution from ∞), and redefining $z \to z/\lambda$, in the limit $\lambda \to \infty$ we get:

$$R(\mathbf{y}, \mathbf{p}) \; = \; \prod_{\alpha=1}^{M} \Bigg[\frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{dz_\alpha}{z_\alpha} \exp\{z_\alpha y_\alpha\} \Bigg] \; \lim_{\lambda\to\infty} \Phi\Big[\mathbf{p}, \; \frac{\mathbf{p}}{\lambda}, \; \mathbf{z}; \; \frac{\mathbf{z}}{\lambda}\Big] \tag{73}$$

where the parameters y_α, p_α and z_α remain finite in the limit $\lambda \to \infty$.

Let us consider now the summations in eq.(69). Here the double sum can be represented as follows:

$$\begin{aligned} \sum_{m_\alpha + s_\alpha \geq 1}^{\infty} (-1)^{m_\alpha + s_\alpha - 1} \, f\big(\mathbf{m}; \mathbf{s}\big) \; = \; \sum_{m_\alpha = 1}^{\infty} (-1)^{m_\alpha - 1} \, f\big(\mathbf{m}; \mathbf{s}\big)\big|_{s_\alpha = 0} \; + \; \sum_{s_\alpha = 1}^{\infty} (-1)^{s_\alpha - 1} \, f\big(\mathbf{m}; \mathbf{s}\big)\big|_{m_\alpha = 0} \; - \\ - \; \sum_{m_\alpha = 1}^{\infty} (-1)^{m_\alpha - 1} \sum_{s_\alpha = 1}^{\infty} (-1)^{s_\alpha - 1} f\big(\mathbf{m}; \mathbf{s}\big) \end{aligned} \tag{74}$$

Thus, according to the above summation algorithm, we get

$$\lim_{\lambda\to\infty} \sum_{m_\alpha + s_\alpha \geq 1}^{\infty} (-1)^{m_\alpha + s_\alpha - 1} \, f\big(\mathbf{m}; \mathbf{s}\big) = \frac{1}{(2\pi i)^2} \int\int_{\mathcal{C}'} \frac{dz_{1\alpha}}{z_{1\alpha}} \frac{dz_{2\alpha}}{z_{2\alpha}} \Big[(2\pi i)\delta(z_{2\alpha}) + (2\pi i)\delta(z_{1\alpha}) - 1\Big] \lim_{\lambda\to\infty} f\Big(\frac{\mathbf{z}_1}{\lambda}; \frac{\mathbf{z}_2}{\lambda}\Big) \tag{75}$$

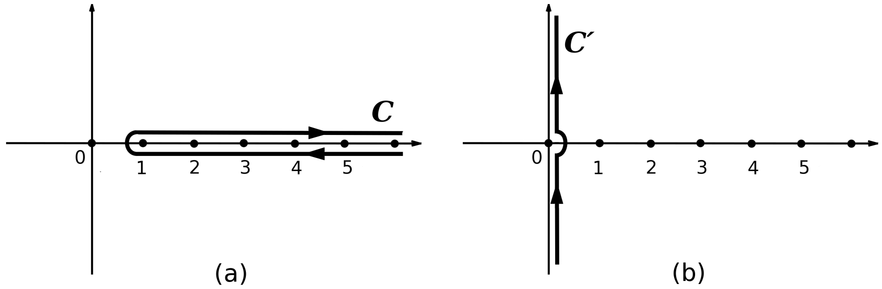

FIG. 1: The contours of integration in the complex plane used for summing the series: (a) the original contour $\mathcal{C}$; (b) the deformed contour $\mathcal{C}'$;

where the rescaled integration parameters $z_{1\alpha}$ and $z_{2\alpha}$ remain finite in the limit $\lambda \to \infty$. Finally, taking into account the Gamma function properties, $\Gamma(z)|_{|z|\to 0} = 1/z$ and $\Gamma(1+z)|_{|z|\to 0} = 1$, for the factors $\mathcal{G}$ and $\mathcal{G}_{\alpha\beta}$, eqs.(61)-(62), we easily find

$$
\begin{aligned}
\lim_{\lambda\to\infty} \mathcal{G}\big(q_\alpha, m_\alpha, s_\alpha\big)\; 2^{(m_\alpha+s_\alpha)} \;&=\; \lim_{\lambda\to\infty} \mathcal{G}\Big(\frac{p_\alpha}{\lambda},\; \frac{z_{1\alpha}}{\lambda},\; \frac{z_{2\alpha}}{\lambda}\Big) 2^{(z_{1\alpha}/\lambda + z_{2\alpha}/\lambda)} \\
&=\; \frac{\big(z_{1\alpha}+z_{2\alpha}+ip_\alpha^{(-)}\big)\big(z_{1\alpha}+z_{2\alpha}-ip_\alpha^{(+)}\big)}{\big(z_{2\alpha}+ip_\alpha^{(-)}\big)\big(z_{1\alpha}-ip_\alpha^{(+)}\big)} \;\equiv\; \mathcal{G}_*\big(p_\alpha, z_{1\alpha}, z_{2\alpha}\big)
\end{aligned}
\tag{76}
$$

and

$$
\lim_{\lambda\to\infty} \mathcal{G}_{\alpha\beta}\Big(\frac{\mathbf{p}}{\lambda},\; \mathbf{m},\; \mathbf{s}\Big) \;=\; \lim_{\lambda\to\infty} \mathcal{G}_{\alpha\beta}\Big(\frac{\mathbf{p}}{\lambda},\; \frac{\mathbf{z}_1}{\lambda},\; \frac{\mathbf{z}_2}{\lambda}\Big) \;=\; 1
\tag{77}
$$

where

$$
p_\alpha^{(\pm)} \;=\; p_\alpha \;\pm\; i\epsilon
\tag{78}
$$

Thus, in the limit $\lambda \to \infty$ the expression for the probability distribution function, eq.(69), takes the form of the Fredholm determinant

$$
\begin{aligned}
W(f) \;&=\; 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{\alpha=1}^{M} \Bigg[\int\int_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{2\pi} \,\mathrm{Ai}\big(y_\alpha + p_\alpha^2 - f\big) \\
&\times\; \frac{1}{(2\pi i)^2} \int\int_{\mathcal{C}'} \frac{dz_{1\alpha}}{z_{1\alpha}} \frac{dz_{2\alpha}}{z_{2\alpha}} \Big[(2\pi i)\delta(z_{2\alpha}) + (2\pi i)\delta(z_{1\alpha}) - 1\Big] \Big(1 + \frac{z_{1\alpha}}{z_{2\alpha}+ip_\alpha^{(-)}}\Big)\Big(1 + \frac{z_{2\alpha}}{z_{1\alpha}-ip_\alpha^{(+)}}\Big)\, \mathrm{e}^{(z_{1\alpha}+z_{2\alpha})y_\alpha} \Bigg] \\
&\times\; \det \hat{K}\big[(z_{1\alpha}, z_{2\alpha}, p_\alpha); (z_{1\beta}, z_{2\beta}, p_\beta)\big]_{\alpha,\beta=1,\dots,M} \\
&=\; \det\big[1 \,-\, \hat{K}\big]
\end{aligned}
\tag{79}
$$

with the kernel

$$
\hat{K}\big[(z_1,\, z_2,\, p); ({z_1}',\, {z_2}',\, p')\big] \;=\; \frac{1}{z_1 + z_2 - ip + {z_1}' + {z_2}' + ip'}
\tag{80}
$$

In the exponential representation of this determinant we get

$$
W(f) \;=\; \exp\Big[-\sum_{M=1}^{\infty} \frac{1}{M}\, \mathrm{Tr}\, \hat{K}^M\Big]
\tag{81}
$$

where

$$\mathrm{Tr}\,\hat{K}^M = \prod_{\alpha=1}^{M}\Bigg[\int\int_{-\infty}^{+\infty}\frac{dy_\alpha dp_\alpha}{2\pi}\,\mathrm{Ai}\big(y_\alpha + p_\alpha^2 - f\big)\times$$
$$\times\;\frac{1}{(2\pi i)^2}\int\int_{\mathcal{C}'}\frac{dz_{1\alpha}}{z_{1\alpha}}\frac{dz_{2\alpha}}{z_{2\alpha}}\Big[(2\pi i)\delta(z_{2\alpha}) + (2\pi i)\delta(z_{1\alpha}) - 1\Big]\Big(1 + \frac{z_{1\alpha}}{z_{2\alpha} + ip_\alpha^{(-)}}\Big)\Big(1 + \frac{z_{2\alpha}}{z_{1\alpha} - ip_\alpha^{(+)}}\Big)\,\mathrm{e}^{(z_{1\alpha}+z_{2\alpha})y_\alpha}\Bigg]\times$$
$$\times\;\prod_{\alpha=1}^{M}\Bigg[\frac{1}{z_{1\alpha} + z_{2\alpha} - ip_\alpha + z_{1\alpha+1} + z_{2\alpha+1} + ip_{\alpha+1}}\Bigg] \tag{82}$$

Here, by definition, it is assumed that $z_{i_{M+1}} \equiv z_{i_1}$ $(i = 1, 2)$ and $p_{M+1} \equiv p_1$. Substituting

$$\frac{1}{z_{1\alpha} + z_{2\alpha} - ip_\alpha + z_{1\alpha+1} + z_{2\alpha+1} + ip_{\alpha+1}} = \int_0^\infty d\omega_\alpha \exp\Big[-\big(z_{1\alpha} + z_{2\alpha} - ip_\alpha + z_{1\alpha+1} + z_{2\alpha+1} + ip_{\alpha+1}\big)\,\omega_\alpha\Big] \tag{83}$$

into eq.(82), we obtain

$$\mathrm{Tr}\,\hat{K}^M = \int_0^\infty d\omega_1 \ldots d\omega_M \prod_{\alpha=1}^{M}\Bigg[\int\int_{-\infty}^{+\infty}\frac{dy_\alpha dp_\alpha}{2\pi}\,\mathrm{Ai}\big(y_\alpha + p_\alpha^2 + \omega_\alpha + \omega_{\alpha-1} - f\big)\exp\{ip_\alpha\big(\omega_\alpha - \omega_{\alpha-1}\big)\}\; S\big(p_\alpha,\, y_\alpha\big)\Bigg] \tag{84}$$

where, by definition, $\omega_0 \equiv \omega_M$, and

$$S\big(p,\, y\big) = \frac{1}{(2\pi i)^2}\int\int_{\mathcal{C}'}\frac{dz_1}{z_1}\frac{dz_2}{z_2}\Big[(2\pi i)\delta(z_2) + (2\pi i)\delta(z_1) - 1\Big]\Big(1 + \frac{z_1}{z_2 + ip^{(-)}}\Big)\Big(1 + \frac{z_2}{z_1 - ip^{(+)}}\Big)\,\mathrm{e}^{(z_1+z_2)y_\alpha} \tag{85}$$

Simple calculations yield:

$$S\big(p,\, y\big) = \frac{1}{2\pi i}\int_{\mathcal{C}'}\frac{dz_1}{z_1}\Big(1 + \frac{z_1}{i(p - i\epsilon)}\Big)\exp\{z_1 y\} + \frac{1}{2\pi i}\int_{\mathcal{C}'}\frac{dz_2}{z_2}\Big(1 - \frac{z_2}{i(p + i\epsilon)}\Big)\exp\{z_2 y\} -$$
$$-\;\frac{1}{(2\pi i)^2}\int\int_{\mathcal{C}'}\frac{dz_1}{z_1}\frac{dz_2}{z_2}\Big(1 + \frac{z_1}{z_2 + i(p - i\epsilon)}\Big)\Big(1 + \frac{z_2}{z_1 - i(p + i\epsilon)}\Big)\exp\{(z_1 + z_2)y\}$$
$$= \Big[\frac{1}{i(p - i\epsilon)} - \frac{1}{i(p + i\epsilon)}\Big]\,\delta(y) \tag{86}$$

Taking the limit $\epsilon \to 0$ we find:

$$S\big(p,\, y\big) = \delta(y)\delta(p) \tag{87}$$

Substituting this result into eq.(84) we obtain

$$\mathrm{Tr}\,\hat{K}^M = \int_0^\infty d\omega_1 \ldots d\omega_M \prod_{\alpha=1}^{M}\Big[\mathrm{Ai}\big(\omega_\alpha + \omega_{\alpha-1} - f\big)\Big] \tag{88}$$

In other words, the free energy distribution function of our problem is given by the Fredholm determinant,

$$W(f) = \det\Big[1 - \hat{K}_{-f}\Big] \tag{89}$$

with the kernel

$$K_{-f}(\omega, \omega') = \mathrm{Ai}\big(\omega + \omega' - f\big)\,, \qquad (\omega,\, \omega' > 0) \tag{90}$$

which is the GOE Tracy-Widom distribution [14, 15].

IV. CONCLUSIONS

In this paper we have presented sufficiently simple derivation of the GOE Tracy-Widom distribution function for the free energy fluctuations in random directed polymers with free boundary conditions. The main message of this somewhat technical work is not the final result itself (which is not new anyway), but the demonstration of the efficiency of the general method and new technical tricks used in the derivation. By mapping the original problem to the N-particle quantum boson system with attractive interactions the derivation is done in the framework of the integer replica series summations and the Bethe ansatz formalism for the quantum boson system.

The key technical tricks of presented calculations includes the following points. First of all, to make the integration over particle coordinates of the Bethe ansatz propagator well defined one has to introduce proper regularization at $\pm\infty$ which requires formal splitting the partition function into two parts: the one in the positive particles coordinates sector (up to $+\infty$) and another one in the negative particles coordinates sector (down to $-\infty$), eqs.(15)-(16). Next is the "magic" Bethe ansatz combinatorial identity, eq.(37), which allows to perform the summation over the momenta permutations and "disentangle" sophisticated products containing in the Bethe ansatz propagator. One more trick is the reformulation of the summation over permutations of the momenta between the positive and the negative particles positions sectors in terms of the series summations, eq.(43), which allows to represent the probability distribution function in terms of the problem of the series summations, eq.(49). Finally, the crucial point of the considered derivation is the procedure of the series summations in the thermodynamic limit $t \to \infty$. In this limit, due to the integral representation of the series, eqs.(71)-(73), one obtains dramatic simplifications of some factors, eqs.(76)-(77), in the expression for the probability distribution function, which allows to represent it in the form of the Fredholm determinant, eq.(79).

Hopefully, the experience obtained in the presented calculations would help to solve more serious long standing problems of this scope, such as the distribution function of the directed polymer's end point fluctuations or the statistical properties of the free energy fluctuations at different times.

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